Theorem on Electromagnetic Backscatter*

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A theorem is proved which gives sufficient conditions under which electromagnetic backscatter from an inhomogeneous object vanishes identically. It is shown that under these conditions the electric and magnetic fields are algebraically related. The special example of a spherically symmetric inhomogeneous scatterer is considered, and the low-frequency differential cross section is derived. An interesting result at high frequencies is also pointed out.

I. INTRODUCTION

THERE is considerable current interest in minimizing radar cross sections. Thus, it seems pertinent to examine the ideal situation of zero electromagnetic backscatter. While the set of both necessary and sufficient conditions for zero backscatter is not known, the following theorem does provide one simple sufficient criterion[1]:

If a plane wave is incident along the axis of symmetry of an axially symmetric scatterer, and if the relative permittivity and permeability of the obstacle satisfy the relation $\epsilon(\mathbf{r})=\mu(\mathbf{r})$, then the radar cross section is identically zero for all frequencies.

The theorem is first proved in its general form, then demonstrated for the important special case of an inhomogeneous spherically symmetric scatterer. The analytical methods of the latter derivation are also used to deduce the angular distribution of low-frequency radiation scattered from such a medium. An interesting result at high frequencies is also pointed out.

II. PROOF OF THE THEOREM

Maxwell's equations, assuming harmonic time dependence, may be written in the form of two stationary wave equations

$$\nabla\times\nabla\times\mathbf{E}-k^2\mathbf{E}+U(\mathbf{r})\mathbf{E}-\frac{\nabla\mu(\mathbf{r})}{\mu(\mathbf{r})}\times\nabla\times\mathbf{E}=0, \quad (1)$$

$$\nabla\times\nabla\times\mathbf{H}-k^2\mathbf{H}+U(\mathbf{r})\mathbf{H}-\frac{\nabla\epsilon(\mathbf{r})}{\epsilon(\mathbf{r})}\times\nabla\times\mathbf{H}=0, \quad (2)$$

where the relative permittivity ϵ and relative permeability μ are arbitrary complex functions of $\mathbf{r}$, and $U(\mathbf{r})\equiv k^2[1-\mu(\mathbf{r})\epsilon(\mathbf{r})]$. The standard boundary conditions for a scattering problem are assumed: At infinity the total fields are the sum of an incident plane wave and an outgoing spherical wave; the usual continuity conditions at surfaces of discontinuity, if any, of $\epsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ are also assumed.

To specialize to the axially symmetric problem, the axis of symmetry and the direction of propagation of the incident plane wave are both chosen to be along the z axis; i.e., $\mathbf{E}_0\times\mathbf{H}_0^*$ is a vector pointing in the $+z$ direction. The assumption of an outgoing scattered wave implies that, in the backward direction, the phase of $\mathbf{H}$ relative to that of $\mathbf{E}$ has been changed so that $\mathbf{E}_{\text{scatt}}\times\mathbf{H}^*_{\text{scatt}}$ is a vector pointing in the $-z$ direction. (The change in relative phase of $\mathbf{E}$ and $\mathbf{H}$ is possible since $\mathbf{E}$ and $\mathbf{H}$ are solutions of different equations.)

Suppose now that $\epsilon(\mathbf{r})\equiv\mu(\mathbf{r})$. The differential equations for $\mathbf{E}$ and $\mathbf{H}$ are then identical, and they may be replaced by the single equation

$$\nabla\times\nabla\times\mathbf{K}(\mathbf{r})-k^2\mathbf{K}(\mathbf{r})+U(\mathbf{r})\mathbf{K}(\mathbf{r})-\frac{\nabla\mu(\mathbf{r})}{\mu(\mathbf{r})}\times\nabla\times\mathbf{K}(\mathbf{r})=0. \quad (3)$$

There are two linearly independent vector solutions to this equation, one corresponding to the incident wave, $\mathbf{K}_0(\mathbf{r})$, polarized in the x direction and the other corresponding to $\mathbf{K}_0(\mathbf{r})$ polarized in the y direction. In the first case, the assumed axial symmetry requires the backscattered field to be polarized in the x direction, while in the second case the backscattered field must be polarized in the y direction. Furthermore, because of the axial symmetry the phase change of the x-polarized backscattered wave must be exactly equal to the phase change of the y-polarized backscattered wave. Therefore, the relative phase of the two scattered waves is the same as their relative phases in the incident wave. Identifying $\mathbf{E}$ with the solution corresponding to the x-polarized incident wave and $\mathbf{H}$ with the solution corresponding to the y-polarized incident wave, we conclude that $\mathbf{E}_{\text{scatt}}\times\mathbf{H}^*_{\text{scatt}}$ must be a vector pointing in the direction of propagation of the incident wave. But this is consistent with the assumption of outgoing scattered waves only if the backscattered fields are identically zero. There are no explicit restrictions on frequency, and the theorem is, therefore, valid for all frequencies for which $\epsilon=\mu$.

Note that the strict equality of $\epsilon(\mathbf{r})$ and $\mu(\mathbf{r})$ is actually not necessary for the validity of the theorem.

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[1] A special case of this result was, to the authors' knowledge, first discussed by V. H. Weston [Conductron Corporation, Report CAA-005-10-P, Appendix H, 1961 (unpublished)] who required $\epsilon=\mu=$const in the scattering medium. The proof offered by Weston for the homogeneous case is essentially the same as that used by the authors in the proof of the general case (Sec. II).

It is clear from Eq. (1) that the $\mathbf{E}$ and $\mathbf{H}$ equations are identical provided only that the equality $\nabla\epsilon/\epsilon=\nabla\mu/\mu$ is satisfied at every space point $\mathbf{r}$. The most general relation between μ and ϵ satisfying this equality is $\epsilon(\mathbf{r})=b\mu(\mathbf{r})$, where b is any constant. Physically, the case when $b\neq1$ represents the problem in which the source of the plane wave, the scatterer, and the observer are all imbedded in a medium whose relative permittivity and/or permeability are different from unity. The proof of the theorem, for the more general case $b\neq1$, proceeds essentially as before, with only a redefinition of the "free-space" wave number required.

III. REMARKS

An interesting consequence of the fact that the $\mathbf{E}$ and $\mathbf{H}$ fields are described by a single vector equation is that there exists an explicit nondifferential relation between the $\mathbf{E}$ and $\mathbf{H}$ fields. It may be shown that if $\mathbf{K}(\mathbf{r})$ is a solution to Eq. (3), then $\mathcal{R}K(\mathcal{R}^{-1}r)$ is also a solution, where $\mathcal{R}$ is the rotation operator (referred to a Cartesian basis):

$$\mathcal{R}=\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}.$$

This may be established by operating on Eq. (3) with $\mathcal{R}$, replacing the argument $\mathbf{r}$ of the scalar and vector functions by $\mathcal{R}^{-1}\mathbf{r}$, and using the invariance property $\mu(\mathcal{R}^{-1}\mathbf{r})=\mu(\mathbf{r})$; a small amount of algebra then shows that Eq. (3) is recovered, but with $\mathcal{R}\mathbf{K}(\mathcal{R}^{-1}\mathbf{r})$ replacing $\mathbf{K}(\mathbf{r})$. Since the incident field $\mathbf{K}_0(r)$ is a plane wave propagating in the $+z$ direction, we may choose $\mathbf{K}_0(\mathbf{r})=\hat{x}e^{ikz}$. Then $\mathcal{R}\mathbf{K}_0(\mathcal{R}^{-1}\mathbf{r})=\hat{y}e^{ikz}$ is a vector representing an incident plane wave polarized in the $+y$ direction. Therefore, if $\mathbf{K}(\mathbf{r})$ is one solution of Eq. (3), $\mathcal{R}\mathbf{K}(\mathcal{R}^{-1}\mathbf{r})$ is the second linearly independent solution, and we may identify $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ with $\mathbf{K}(\mathbf{r})$ and $\mathcal{R}\mathbf{K}(\mathcal{R}^{-1}\mathbf{r})$, respectively.[2] This identification is consistent with the boundary conditions on $\mathbf{E}$ and $\mathbf{H}$ at the surface of the scatterer, for the requirement that $\hat{n}(\mathbf{r})\times\mathbf{K}(\mathbf{r})$ be continuous transforms into the condition that $\hat{n}(\mathbf{r})\times\mathcal{R}\mathbf{K}(\mathcal{R}^{-1}\mathbf{r})$ be continuous.

Incidentally, the relation $\mathbf{H}(\mathbf{r})=\mathcal{R}\mathbf{E}(\mathcal{R}^{-1}\mathbf{r})$ can now be used to give a very simple proof of the theorem, for on the z-axis $\mathcal{R}\mathbf{E}(\mathcal{R}^{-1}\mathbf{r})=\mathcal{R}\mathbf{E}(\mathbf{r})$, so that the z component of $\mathbf{S}=\mathbf{E}\times\mathbf{H}^*$ becomes simply

$$S_z(0,0,z)=|E_x(0,0,z)|^2+|E_y(0,0,z)|^2.$$

Since $S_z(0,0,z)\geq0$, the far-zone scattered field on the symmetry axis must propagate only in the $+z$ direction, which contradicts the outgoing-wave boundary condition in the backward direction, unless the scattered field is zero.

Since the backscatter cross section is zero when $\epsilon(\mathbf{r})=\mu(\mathbf{r})$, it should increase continuously from zero as $\epsilon(\mathbf{r})-\mu(\mathbf{r})$ is allowed to differ slightly from zero everywhere. This suggests that there may exist an expansion of the fields in terms of a uniformly small quantity, $f[\epsilon(\mathbf{r})]-f[\mu(\mathbf{r})]$, which should hold for large, as well as small, values of ϵ. In any case, the fact that the cross section in the backward direction must vanish when $\epsilon=\mu$, should serve as an additional validity criterion for any approximation method developed to apply when ϵ and μ both differ from unity.

IV. THE SPHERICALLY SYMMETRIC CASE

It would be useful if the angular distribution of the radiation, when $\mu=\epsilon$, could be compared with that when $\epsilon\neq\mu=1$ in order to determine whether the radiation which is not scattered in the backward direction appears instead at angles close to π, or whether the forward scattering amplitude is enhanced. Such a comparison is not possible for the general case. However, it is now shown that, for long wavelengths, the angular distribution for a spherically symmetric, but inhomogeneous, scatterer has a particularly simple form when $\epsilon(r)=\mu(r)$, and that the radiation pattern is peaked in the forward direction.

The theory of scattering from an inhomogeneous sphere has been developed by several authors[3–5] for the special case $\mu=1$. The following expression for the electric field is the generalization to a system with varying $\mu(r)$, as can be verified by direct substitution into the Maxwell equations,

$$\epsilon(r)\mathbf{E}(\mathbf{r})=\nabla\times[\mu^{1/2}(r)\epsilon(r)\psi(\mathbf{r})\mathbf{r}]+\frac{1}{k}\nabla\times\{\nabla\times[\epsilon^{1/2}(r)\phi(\mathbf{r})\mathbf{r}]\},\quad(4)$$

$$ik\mathbf{B}(\mathbf{r})=\nabla\times\mathbf{E}(\mathbf{r}),\quad(5)$$

where ψ and ϕ satisfy the following equations:

$$\nabla^2\psi+\left[k^2\mu\epsilon-\mu^{1/2}\frac{d^2}{dr^2}(\mu^{-1/2})\right]\psi=0,\quad(6)$$

$$\nabla^2\phi+\left[k^2\mu\epsilon-\epsilon^{1/2}\frac{d^2}{dr^2}(\epsilon^{-1/2})\right]\phi=0.\quad(7)$$

The boundary conditions on ψ and ϕ must be such that

$$\mathbf{E}(\mathbf{r})\underset{r\to\infty}{\longrightarrow}\hat{x}\exp(ikz)+\mathbf{A}(\theta,\phi)r^{-1}\exp(ikr).\quad(8)$$

Here, $\hat{x}$ is the initial polarization, and $\mathbf{A}$ is the vector scattering amplitude. The absolute square of $\mathbf{A}$ is the differential cross section.

[2] This is true in Gaussian units, but because of the linearity of the equations we may make the more general statement that $\mathbf{H}(\mathbf{r})=h\mathcal{R}\mathbf{E}(\mathcal{R}^{-1}\mathbf{r})$, where h is a constant appropriate to the chosen system of units.

[3] C. T. Tai, Appl. Sci. Res. Sect. **B7**, 113 (1958).
[4] D. Arnush, Space Technology Laboratories, Inc., Report No. 6110-7466-RU-001 (unpublished).
[5] P. J. Wyatt, Phys. Rev. **127**, 1837 (1962).

The radial equations associated with Eqs. (6) and (7) are

$$\frac{d^2}{dr^2}(rR_l)+\left[k^2\mu\epsilon-\mu^{1/2}\frac{d^2}{dr^2}(\mu^{-1/2})-\frac{l(l+1)}{r^2}\right]rR_l=0,$$

$$\frac{d^2}{dr^2}(rS_l)+\left[k^2\mu\epsilon-\epsilon^{1/2}\frac{d^2}{dr^2}(\epsilon^{-1/2})-\frac{l(l+1)}{r^2}\right]rS_l=0,$$

with boundary conditions

$$rR_l,\quad rS_l \underset{r\to 0}{\longrightarrow} 0,$$

$$rR_l \underset{r\to\infty}{\longrightarrow} \sin(kr-l\pi/2+\delta_l),$$

$$rS_l \underset{r\to\infty}{\longrightarrow} \sin(kr-l\pi/2+\eta_l).$$

The phase shifts, δ_l and η_l, determine the scattering. When $\mu(\mathbf{r})=\epsilon(\mathbf{r})$, the radial equations are identical, and $\delta_l=\eta_l$.

The scattering amplitude is derived by substituting expansions of the form

$$\sum\nolimits_{l,m} a_{l,m}R_l(r)Y_l^m(\theta,\phi)$$

for ψ and ϕ in Eq. (4). The expansion coefficients can then be evaluated by imposing the asymptotic condition, Eq. (8), provided that the vector plane wave is expressed by its known expansion[6] in spherical harmonics. In general, $\mathbf{A}$ is a complicated function of angles, but because of the equality of the phase shifts δ_l and η_l when $\mu(r)=\epsilon(r)$, considerable simplification of the vector-scattering amplitude is possible. It is readily shown that in this case $\mathbf{A}(\theta,\phi)$ reduces to the relatively simple expression

$$\mathbf{A}(\theta,\phi)=(2ik)^{-1}(\cos\phi\hat{\theta}-\sin\phi\hat{\Phi})\sum_{l=1}^{\infty}\frac{(2l+1)}{l(l+1)}(e^{2i\delta_l}-1)$$
$$\times\left[(1-\tau)\frac{dP_l(\tau)}{d\tau}+l(l+1)P_l(\tau)\right],\quad (9)$$

where $\tau\equiv\cos\theta$. The theorem can now be easily verified for this special case since, for $\theta=\pi$, the quantity in square brackets vanishes for every value of l, and $\mathbf{A}(\pi)$ is, therefore, identically zero.

When $ka\ll 1$, where a is the characteristic dimension of the scatterer, only the $l=1$ phase shift is important. One then finds from Eq. (9) that

$$|\mathbf{A}(\theta,\phi)|^2=\frac{9}{4k^2}\sin^2\delta_1(1+\cos\theta)^2,\quad (10)$$

in contrast to a $(1+\cos^2\theta)$ angular dependence of the differential cross section when $\mu=1$.[7] Thus, at least in the long-wavelength limit, the distribution shifts to predominantly forward scattering.

In the short-wavelength limit, the Schiff approximation for large-angle electromagnetic scattering[8] can be used to compute the angular distribution in the neighborhood of the backward direction. With his choice of axes, this approximation takes the following form when $\mu=\epsilon$:

$$\mathbf{A}(\hat{k}_f,\hat{k}_0)\simeq-\frac{k^2}{4\pi}(1+\cos\theta)$$
$$\times(\cos\theta\cos\phi\,\hat{x}+\sin\phi\,\hat{y}-\sin\theta\cos\phi\,\hat{z})$$
$$\times\int d\mathbf{r}[1-\epsilon(\mathbf{r})]\exp\left\{i\mathbf{q}\cdot\mathbf{r}+\frac{ik}{2}\right.$$
$$\left.\times\left[\int_0^\infty B(\mathbf{r}-\hat{k}_0s)ds+\int_0^\infty B(\mathbf{r}+\hat{k}_fs)ds\right]\right\},\quad (11)$$

where

$$B(\mathbf{r})=\epsilon^2(\mathbf{r})-1,\qquad \mathbf{q}=\mathbf{k}_0-\mathbf{k}_f.\quad (12)$$

The unit vectors $\hat{k}_0$ and $\hat{k}_f$ point in the initial and final directions, respectively. Equation (11) is zero in the backward direction, under no assumptions other than $B\ll 1$, $kR\gg 1$, where R is a characteristic dimension of the scatterer. The assumption of axial symmetry is not required. For angles close to 180°, the angular dependence of the cross section is again approximately $(1+\cos\theta)^2$. This follows from a Taylor expansion of the integral in Eq. (11) about $\theta=\pi$. When $\epsilon(\mathbf{r})=\epsilon(r)$, this approximation to the angular dependence is even better since the linear term in the Taylor expansion is zero.

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[6] W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), p. 205.

[7] The result for $\mu=1$ refers to the less general case of a homogeneous sphere, in which the differential cross section has been integrated over ϕ. See J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 570.

[8] L. I. Schiff, Phys. Rev. **104**, 1481 (1956).